AP Calculus Logistic Growth Homework Name: Key

1. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is time in years. Find $\lim_{t\to\infty} P(t)$.

If $\frac{dy}{dt} = ky(L-y)$, then $\frac{dP}{dt} = kP(L-P) \rightarrow$

Since $\frac{dP}{dt} = P(2-\frac{P}{5000}) \rightarrow$ so $\frac{dP}{dt} = \frac{1}{5000}P(10000-P)$

From equation, L (carrying capacity) is equal to 10,000, so $\lim_{t\to\infty} P(t)$ tends to L, so $\boxed{= 10,000}$

2. A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$ where t is measured in years.

a) If $P(0) = 20$, find P as a function of t.

$y = \frac{L}{1+ce^{-Lkt}}$

$\frac{dP}{dt} = .01P(100-P)$ (correct form already)

$\downarrow$ k $\downarrow$ L

$P(t) = \frac{100}{1+ce^{-100(.01)t}}$; $20 = \frac{100}{1+ce^{-1(0)}} \rightarrow$ $20(1+c) = 100$

$20 + 20c = 100$

$20c = 80$

$c = 4$

Final: $P(t) = \frac{100}{1+4e^{-1t}}$

b) Use your answer from (a) to find P when $t = 3$ years.

$P(3) = \frac{100}{1+4e^{-3}}$ $\boxed{\approx 83.393 \text{ animals}}$

c) Use your answer from (a) to find t when $P = 80$ animals.

$80 = \frac{100}{1+4e^{-t}} \rightarrow 80 + 320e^{-t} = 100 \rightarrow 320e^{-t} = 20 \rightarrow$

$e^{-t} = \frac{20}{320} \rightarrow -t = \ln\left(\frac{20}{320}\right) \rightarrow \boxed{t \approx 2.773 \text{ yrs}}$

3. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t, in years.

a) Find $\lim_{t\to\infty} P(t)$.

$\frac{dP}{dt} = .01P(300-P)$ L = 300

and $\lim_{t\to\infty} P(t) = L = \boxed{300}$

b) Find the number of wolves in the population when the population is growing the fastest.

this is always when $P(t) = \frac{L}{2}$, so $\frac{300}{2} = \boxed{150}$; $\frac{d^2P}{dt^2} = 0$ and changes + to −

c) When there are 200 wolves in the population, is the rate of change of the population increasing or decreasing? Justify your answer.

200 is more than half, so the rate of change is now decreasing ; $\frac{d^2P}{dt^2} < 0$ since $P > 150$

$$\frac{2x}{(x+1)(x+2)}$$

$$\frac{A}{x+1}+\frac{B}{x+2}=\frac{2x}{(x+1)(x+2)}$$

4. Evaluate the following integrals.

a) $\displaystyle\int \frac{5}{(x-1)(x+2)}\,dx$

$$\frac{5}{3}\int\frac{1}{x-1}dx-\frac{5}{3}\int\frac{1}{x+2}dx$$

$$\frac{A}{x-1}+\frac{B}{x+2}=\frac{5}{(x-1)(x+2)}$$

$$\boxed{=\frac{5}{3}\ln|x-1|-\frac{5}{3}\ln|x+2|+C}$$

$A(x+2)+B(x-1)=5$

$x\to -2 \to -3B=5 \quad B=-5/3$

$x\to 1 \to 3A=5 \quad A=5/3$

b) $\displaystyle\int \frac{2x}{x^2+3x+2}\,dx$

$A(x+2)+B(x+1)=2x$

$x\to -2 \quad -B=-4 \quad B=4$

$x\to -1 \quad A=-2$

$$\int\frac{-2}{x+1}dx+\int\frac{4}{x+2}dx$$

$$\boxed{=-2\ln|x+1|+4\ln|x+2|+C}$$

5. Determine whether the integral converges or diverges, and evaluate the integral if it converges.

a) $\displaystyle\int_4^\infty \frac{-2x}{\sqrt[3]{9-x^2}}\,dx$

$$\lim_{b\to\infty}\int_4^b\frac{-2x}{\sqrt[3]{9-x^2}}dx$$

$u=9-x^2 \quad du=-2x\,dx$

$$\int\frac{1}{\sqrt[3]{u}}du\to\int u^{-\frac{1}{3}}du\to\frac{3}{2}u^{2/3}+C$$

$$\lim_{b\to\infty}\left[\frac{3}{2}(9-x^2)^{2/3}\right]_4^b\to\lim_{b\to\infty}\left(\frac{3}{2}(9-b^2)^{2/3}-\frac{3}{2}(9-4^2)^{2/3}\right)\to\infty-\frac{3}{2}\cdot\sqrt[3]{49}$$

$\boxed{\text{diverges}}$

b) $\displaystyle\int_1^\infty \frac{3}{(5x+1)^2}\,dx$

$u=5x+1 \quad \frac{du}{5}=\frac{5dx}{5}$

$$\lim_{b\to\infty}\int_1^b\frac{3}{(5x+1)^2}dx$$

$$\frac{3}{5}\int\frac{1}{u^2}du\to\frac{-3}{5u}+C$$

$$\lim_{b\to\infty}\left[\frac{-3}{5(5x+1)}\right]_1^b\to\lim_{b\to\infty}\left(\frac{-3}{5(5b+1)}+\frac{3}{5(5+1)}\right)\to\frac{3}{30}=\boxed{\frac{1}{10}}$$

6. Use one of the comparison tests to determine whether $\int_1^\infty \frac{1}{2e^x-5}dx$ converges or diverges.

$$\lim_{x\to\infty}\frac{\frac{1}{2e^x-5}}{\frac{1}{e^x}}\to\frac{e^x}{2e^x-5}\to\frac{1}{2} \quad 0<\frac{1}{2}<\infty$$

since $\int_1^\infty\frac{1}{e^x}dx$ converges, by LCT $\int_1^\infty\frac{1}{2e^x-5}dx$ also converges

$$\int_1^\infty\frac{1}{e^x}dx\to\lim_{b\to\infty}\int_1^b e^{-x}dx\to\lim_{b\to\infty}\left[-e^{-x}\right]_1^b\to\lim_{b\to\infty}(-e^{-b}+e^{-1})\to 0+\frac{1}{e}\to\frac{1}{e}$$

7. Use one of the comparison tests to determine whether $\int_1^\infty \frac{1}{x^4-5x+2}dx$ converges or diverges.

$$\lim_{x\to\infty}\frac{\frac{1}{x^4-5x+2}}{\frac{1}{x^4}}\to\frac{x^4}{x^4-5x+2}\to 1 \quad 0<1<\infty$$

$\int_1^\infty\frac{1}{x^4}dx$ is p-series $a=1$, $p>1$ so converges

$\boxed{\text{by LCT } \int_1^\infty\frac{1}{x^4-5x+2}dx \text{ also converges}}$

8. Evaluate the following integrals.

a) $\displaystyle\int 3\ln x\,dx$

$u=\ln x \quad dv=dx$

$3\int\ln x\,dx \quad du=\frac{1}{x}dx \quad v=x$

$$3\left(x\ln x-\int x\cdot\frac{1}{x}dx\right)$$

$$\boxed{3x\ln x-3x+C}$$

b) $\displaystyle\int 5x\sec(x^2)\tan(x^2)\,dx$

$u=x^2 \quad \frac{du}{2}=\frac{2x\,dx}{2}$

$$5\int x\sec(x^2)\tan(x^2)dx$$

$$\frac{5}{2}\int\sec u\tan u\,du\to\frac{5}{2}\sec u+C$$

$$\boxed{\to\frac{5}{2}\sec(x^2)+C}$$